28.1 The market price of risk

Derivatives depend on a single underlying variable

We start by considering the properties of derivatives dependent on the value of a single variable θ. Assume that the process followed by θ is

$$\frac{d\theta}{\theta} = m\,dt + s\,dz \tag{28.1}$$

- θ : not necessarily the price of a traded security
- m : expected growth rate in θ
- s : volatility of θ

Derivatives depend on a single underlying variable

Suppose that the processes followed by f_1 and f_2 are

$$\frac{df_1}{f_1} = \mu_1\, dt + \sigma_1\, dz$$

and

$$\frac{df_2}{f_2} = \mu_2\, dt + \sigma_2\, dz$$

where μ_1, μ_2, σ_1, and σ_2 are functions of θ and t. The "dz" in these processes must be the same dz as in equation (28.1) because it is the only source of the uncertainty in the prices of f_1 and f_2.

- f_1, f_2：price of two derivatives dependent only on θ $and\ t$
- σ：暴露在風險中的數量

Forming a riskless portfolio

The prices f_1 and f_2 can be related using an analysis similar to the Black–Scholes analysis described in Section 15.6. The discrete versions of the processes for f_1 and f_2 are

$$\Delta f_1 = \mu_1 f_1\,\Delta t + \sigma_1 f_1\,\Delta z \quad \textbf{(28.2)}$$

$$\Delta f_2 = \mu_2 f_2\,\Delta t + \sigma_2 f_2\,\Delta z \quad \textbf{(28.3)}$$

We can eliminate the Δz by forming an instantaneously riskless portfolio consisting of $\sigma_2 f_2$ of the first derivative and $-\sigma_1 f_1$ of the second derivative. If Π is the value of the portfolio, then

$$\Pi = (\sigma_2 f_2)f_1 - (\sigma_1 f_1)f_2 \quad \textbf{(28.4)}$$

and

$$\Delta\Pi = \sigma_2 f_2\,\Delta f_1 - \sigma_1 f_1\,\Delta f_2$$

Substituting from equations (28.2) and (28.3), this becomes

$$\Delta\Pi = (\mu_1\sigma_2 f_1 f_2 - \mu_2\sigma_1 f_1 f_2)\Delta t \quad \textbf{(28.5)}$$

$$\begin{aligned}\Delta\Pi &= \sigma_2 f_2 \Delta f_1 - \sigma_1 f_1 \Delta f_2 \\ &= \sigma_2 f_2 (f_1 \mu_1 \Delta t + f_1 \sigma_1 \Delta z) - \sigma_1 f_1 (f_2 \mu_2 \Delta t + f_2 \sigma_2 \Delta z) \\ &= (\mu_1 \sigma_2 f_1 f_2 - \mu_2 \sigma_1 f_1 f_2)\Delta t\end{aligned}$$

Market price of risk

$(\mu_1 \sigma_2 f_1 f_2 - \mu_2 \sigma_1 f_1 f_2)\Delta t = r(\sigma_2 f_2 f_1 - \sigma_1 f_1 f_2)\Delta t$

$\Rightarrow \mu_1 \sigma_2 f_1 f_2 - \mu_2 \sigma_1 f_1 f_2 = r\sigma_2 f_1 f_2 - r\sigma_1 f_1 f_2$

$\Rightarrow \mu_1 \sigma_2 - \mu_2 \sigma_1 = r\sigma_2 - r\sigma_1$

Because the portfolio is instantaneously riskless, it must earn the risk-free rate. Hence,

$$\Delta\Pi = r\Pi\,\Delta t$$

Substituting into this equation from equations (28.4) and (28.5) gives

$$\Pi = (\sigma_2 f_2)f_1 - (\sigma_1 f_1)f_2 \quad \textbf{(28.4)}$$

$$\Delta\Pi = (\mu_1\sigma_2 f_1 f_2 - \mu_2\sigma_1 f_1 f_2)\Delta t \quad \textbf{(28.5)}$$

$$\mu_1\sigma_2 - \mu_2\sigma_1 = r\sigma_2 - r\sigma_1$$

or

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} \quad \textbf{(28.6)}$$

- $\mu_1 - r$：$risk\ premium$
- σ_1：暴露在風險中的數量
- $\frac{\mu_1 - r}{\sigma_1}$：$f_1$的單位風險溢酬

- 因暴露在相同風險來源，所以投資人要求的單位風險溢酬相同

Market price of risk

Dropping subscripts, equation (28.6) shows that if f is the price of a derivative dependent only on θ and t with

$$\frac{df}{f} = \mu\, dt + \sigma\, dz \tag{28.7}$$

then

$$\frac{\mu - r}{\sigma} = \lambda \tag{28.8}$$

- We refer to $(\mu - r)/\sigma$ as the **market price of risk** for θ and denote it by λ
- If $\lambda\sigma > 0$, investors require a higher return to compensate them for the risk arising from θ.

Market price of risk

Equation (28.8) can be written

$$\mu - r = \lambda\sigma \qquad \textbf{(28.9)}$$

so that

$$df = (r + \lambda\sigma)\, f\, dt + \sigma f\, dz \qquad \textbf{(28.10)}$$

- The market price of risk of a variable determines the growth rate of all securities dependent on the variable.

Traditional risk-neutral world

The process followed by derivative price f is

$$df = \mu f\, dt + \sigma f\, dz$$

The value of μ depends on the risk preferences of investors. In a world where the market price of risk is zero, λ equals zero. From equation (28.9) $\mu = r$, so that the process followed by f is

$$df = r f\, dt + \sigma f\, dz$$

We will refer to this as the *traditional risk-neutral world*.

Equation (28.8) can be written

$$\mu - r = \lambda\sigma \quad \textbf{(28.9)}$$

Girsanov’s theorem

- As we move from one market price of risk to another, the expected growth rates of security prices change, but their volatilities remain the same.

28.2 Several state variables

Several underlying variables

Suppose that n variables, $\theta_1, \theta_2, \ldots, \theta_n$, follow stochastic processes of the form

$$\frac{d\theta_i}{\theta_i} = m_i\,dt + s_i\,dz_i \tag{28.11}$$

for $i = 1, 2, \ldots, n$, where the dz_i are Wiener processes. The parameters m_i and s_i are expected growth rates and volatilities and may be functions of the θ_i and time.

$$\frac{df}{f} = \mu\,dt + \sum_{i=1}^{n} \sigma_i\,dz_i \tag{28.12}$$

In this equation, μ is the expected return from the security and $\sigma_i\,dz_i$ is the component of the risk of this return attributable to θ_i. Both μ and the σ_i are potentially dependent on the θ_i and time.

$$\mu - r = \sum_{i=1}^{n} \lambda_i \sigma_i \tag{28.13}$$

where λ_i is the market price of risk for θ_i.

- n種風險溢酬相加=總風險溢酬

28.3 Martingale

Martingale

A *martingale* is a zero-drift stochastic process.[3] A variable θ follows a martingale if its process has the form

$$d\theta = \sigma\, dz$$

where dz is a Wiener process. The variable σ may itself be stochastic. It can depend on θ and other stochastic variables. A martingale has the convenient property that its expected value at any future time is equal to its value today. This means that

$$E(\theta_T) = \theta_0$$

The equivalent martingale measure result

- If we set λ equal to the volatility of a security g, then Ito's lemma shows that $f\,/\,g$ is a martingale for all derivative security prices f

$$df = (r + \lambda\sigma) f\,dt + \sigma f\,dz \qquad (28.10)$$

$$df = (r + \sigma_g\sigma_f) f\,dt + \sigma_f f\,dz$$

$$dg = (r + \sigma_g^2) g\,dt + \sigma_g g\,dz$$

Using Itô's lemma gives

$$d\ln f = (r + \sigma_g\sigma_f - \sigma_f^2/2)\,dt + \sigma_f\,dz$$

$$d\ln g = (r + \sigma_g^2/2)\,dt + \sigma_g\,dz$$

so that

$$d(\ln f - \ln g) = (\sigma_g\sigma_f - \sigma_f^2/2 - \sigma_g^2/2)\,dt + (\sigma_f - \sigma_g)\,dz$$

or

$$d\left(\ln\frac{f}{g}\right) = -\frac{(\sigma_f - \sigma_g)^2}{2}\,dt + (\sigma_f - \sigma_g)\,dz$$

$$d\ln f = \frac{1}{f}df - \frac{1}{2}\frac{1}{f^2}\sigma_f^2 f^2 dt$$

$$= \frac{1}{f}[(r+\sigma_g\sigma_f) f dt + \sigma_f f dz] - \frac{1}{2}\sigma_f^2 dt$$

$$= (r + \sigma_g\sigma_f - \frac{1}{2}\sigma_f^2) dt + \sigma_f dz$$

$$d\ln g = \frac{1}{g}dg - \frac{1}{2}\frac{1}{g^2}\sigma_g^2 g^2 dt$$

$$= \frac{1}{g}[(r+\sigma_g^2) g dt + \sigma_g g dz] - \frac{1}{2}\sigma_g^2 dt$$

$$= (r + \frac{1}{2}\sigma_g^2) dt + \sigma_g dz$$

The equivalent martingale measure result

- If we set λ equal to the volatility of a security g, then Ito's lemma shows that $f\,/\,g$ is a martingale for all derivative security prices f

$$d\left(\ln\frac{f}{g}\right) = -\frac{(\sigma_f-\sigma_g)^2}{2}\,dt + (\sigma_f-\sigma_g)\,dz$$

$$d\left(\frac{f}{g}\right) = (\sigma_f-\sigma_g)\frac{f}{g}\,dz \qquad \textbf{(28.14)}$$

This shows that f/g is a martingale and proves the equivalent martingale measure result.

$$\frac{f_0}{g_0} = E_g\left(\frac{f_T}{g_T}\right)$$

or

$$f_0 = g_0 E_g\left(\frac{f_T}{g_T}\right) \qquad \textbf{(28.15)}$$

where E_g denotes the expected value in a world defined by numeraire g.

Let $X(t) = \ln\left(\frac{f}{g}\right)$, $Y(t) = \exp(X(t)) = \frac{f}{g}$

$$dY(t) = \frac{\partial Y}{\partial X}dX + \frac{1}{2}\frac{\partial^2 Y}{\partial X^2}(dX)^2$$

$$= \exp(X(t))\,dX(t) + \frac{1}{2}\exp(X(t))(\sigma_f-\sigma_g)^2 dt$$

$$= Y(t)\times\left[-\frac{(\sigma_f-\sigma_g)^2}{2}dt + (\sigma_f-\sigma_g)\,dz\right] + \frac{1}{2}Y(t)(\sigma_f-\sigma_g)^2 dt$$

$$= Y(t)(\sigma_f-\sigma_g)\,dz$$

$$\Rightarrow d\left(\frac{f}{g}\right) = (\sigma_f-\sigma_g)\frac{f}{g}\,dz$$

28.4 Alternative choice for the numeraire

Zero-coupon bond price as the numeraire

Define $P(t, T)$ as the price at time t of a risk-free zero-coupon bond that pays off \$1 at time T. We now explore the implications of setting the numeraire g equal to $P(t, T)$. Let E_T denote expectations in a world defined by this numeraire. Because $g_T = P(T, T) = 1$ and $g_0 = P(0, T)$, equation (28.15) gives

$$f_0 = g_0 E_g\left(\frac{f_T}{g_T}\right) \qquad \textbf{(28.15)}$$

$$f_0 = P(0, T) E_T(f_T) \qquad \textbf{(28.20)}$$

Consider any variable θ that is not an interest rate.[6] A forward contract on θ with maturity T is defined as a contract that pays off $\theta_T - K$ at time T, where θ_T is the value θ at time T. Define f as the value of this forward contract. From equation (28.20),

$$f_0 = P(0, T)[E_T(\theta_T) - K]$$

$$f_0 = P(0,T)E_T(\theta_T - K)$$
$$= P(0,T)[E_T(\theta_T) - K]$$

Zero-coupon bond price as the numeraire

$$f_0 = P(0, T)[E_T(\theta_T) - K]$$

The forward price, F, of θ is the value of K for which f_0 equals zero. It therefore follows that

$$P(0, T)[E_T(\theta_T) - F] = 0$$

or

$$F = E_T(\theta_T) \qquad \textbf{(28.21)}$$

- The forward price of any variable is its expected future spot price in a world defined by the numeraire P(t, T)